Probability and Random Processes

ECS 315

Asst. Prof. Dr. Prapun Suksompong

prapun@siit.tu.ac.th

10 Continuous Random Variables

Office Hours:

BKD, 6th floor of Sirindhralai building

Wednesday	14:30-15:30
Friday	14:30-15:30

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10.1 Probability Density Function

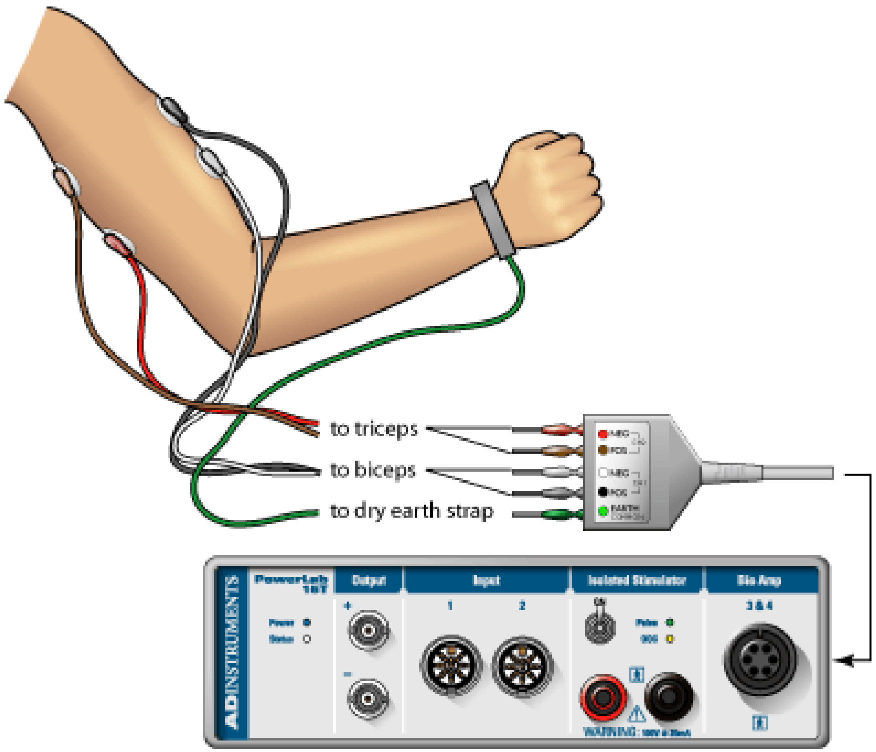
Ex. `rand` function

- Generate an array of uniformly distributed pseudorandom numbers.
  - The pseudorandom values are drawn from the **standard uniform distribution** on the open **interval (0,1)**.
- `rand` returns a scalar.
- `rand(m,n)` or `rand([m,n])` returns an *m*-by-*n* matrix.
  - `rand(n)` returns an *n*-by-*n* matrix

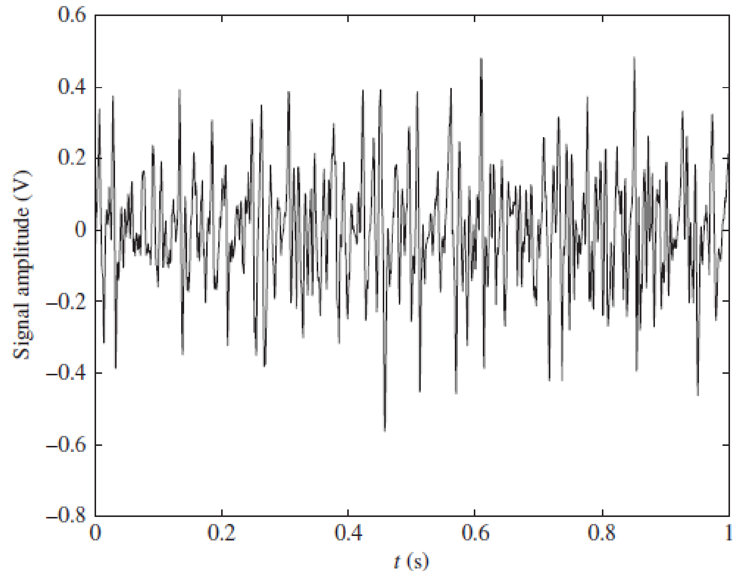
```
>> rand

ans =

    0.3816

>> rand(10,2)

ans =

    0.7655    0.6551
    0.7952    0.1626
    0.1869    0.1190
    0.4898    0.4984
    0.4456    0.9597
    0.6463    0.3404
    0.7094    0.5853
    0.7547    0.2238
    0.2760    0.7513
    0.6797    0.2551
```

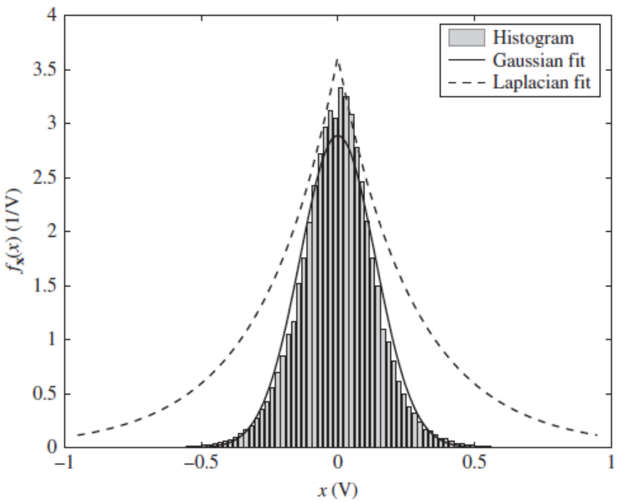
Ex. Muscle Activity

- Look at electrical activity of skeletal muscle by recording a human electromyogram (EMG).



[http://www.adinstruments.com/solutions/education/ltexp/electromyography-emg-german]

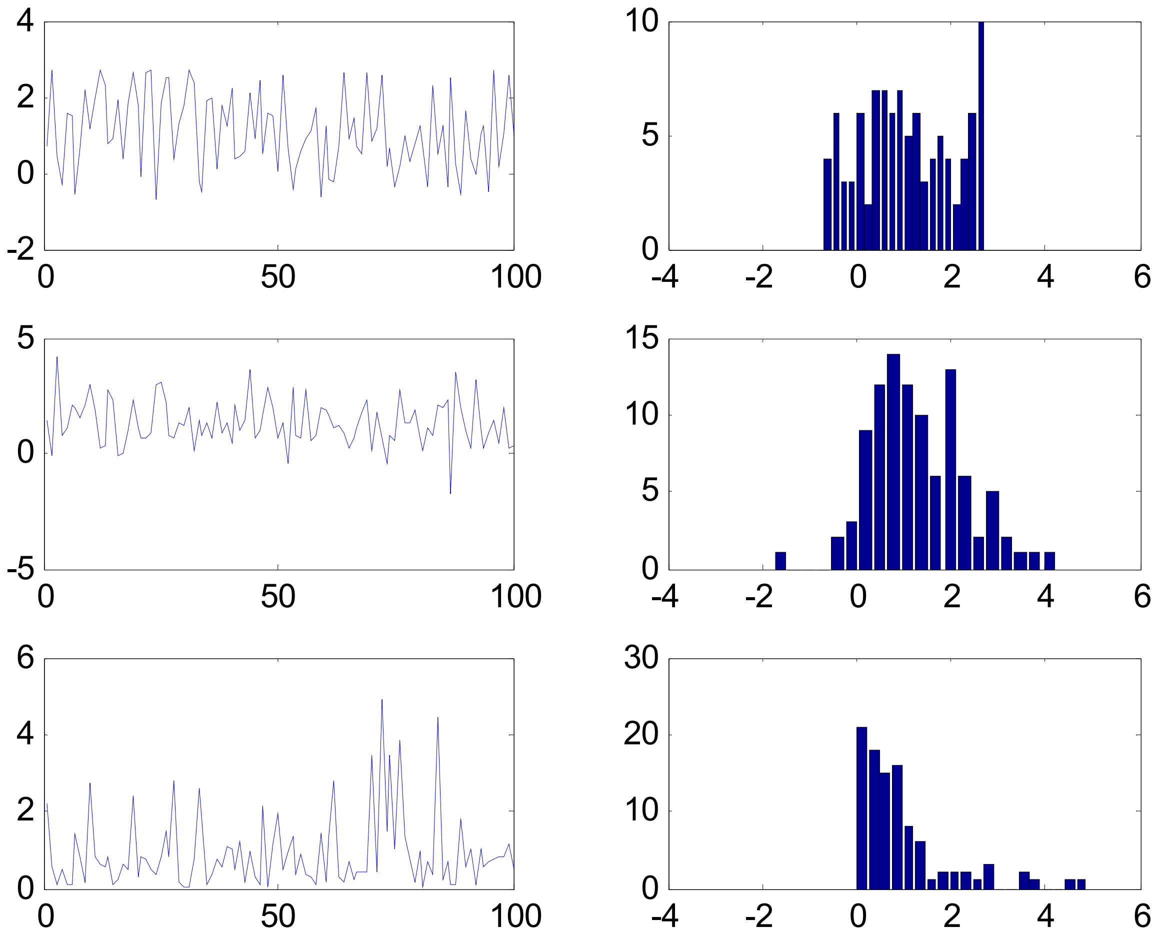




Three Important Continuous RVs

```
close all; clear all;
N = 1e6; b = 20; m = 1; s = 1;
R = [1-5*s,1+5*s];
% Uniform
X = (2*sqrt(3)*(rand(1,N)-0.5))+1;
subplot(3,2,1); plot(X);
subplot(3,2,2); plotHistPdf(X,b)
xlim(R)
% Normal
X = randn(1,N)+1;
subplot(3,2,3); plot(X);
subplot(3,2,4); plotHistPdf(X,b)
xlim(R)
% Exponential
X = exprnd(1,1,N);
subplot(3,2,5); plot(X);
subplot(3,2,6); plotHistPdf(X,b)
xlim(R)
```

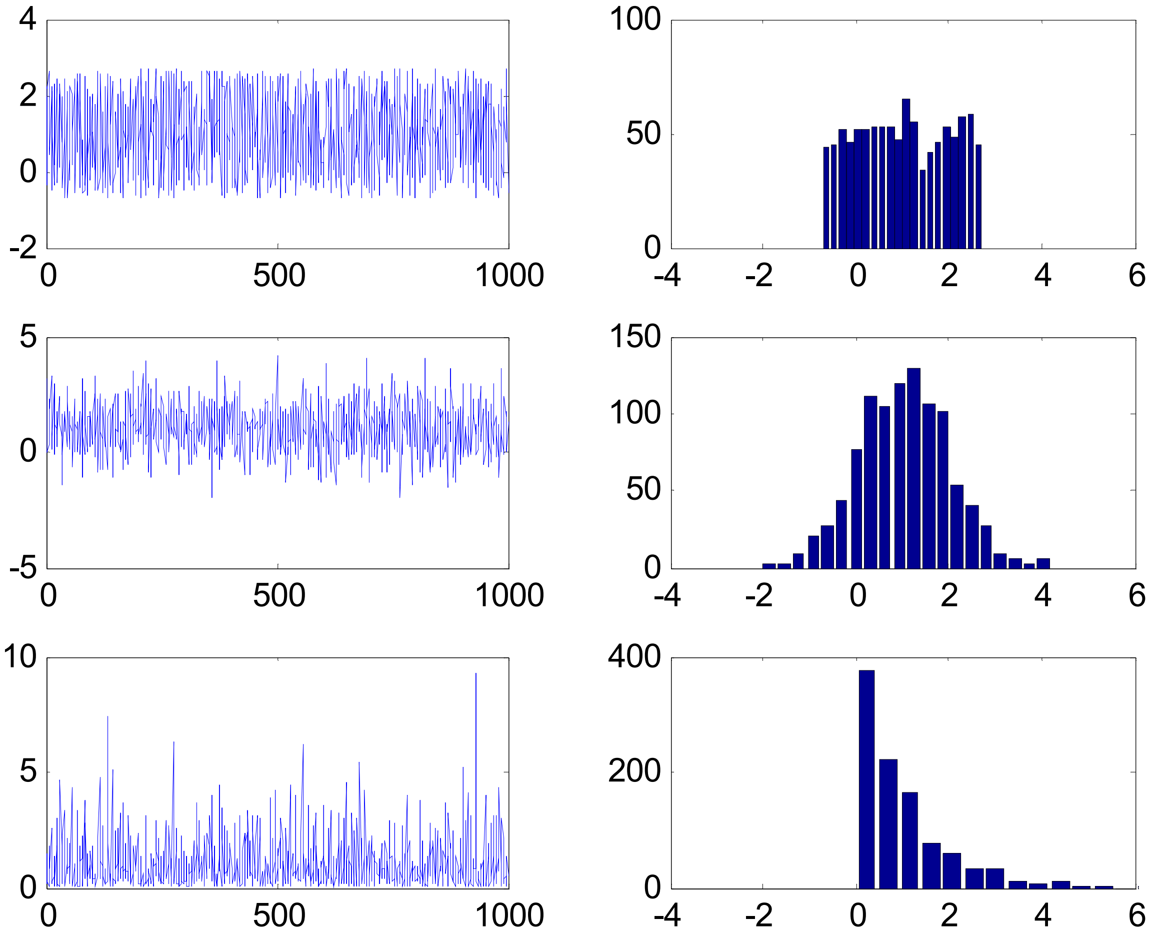
Three Important Continuous RVs



Mean = 1
Std = 1
N = 100

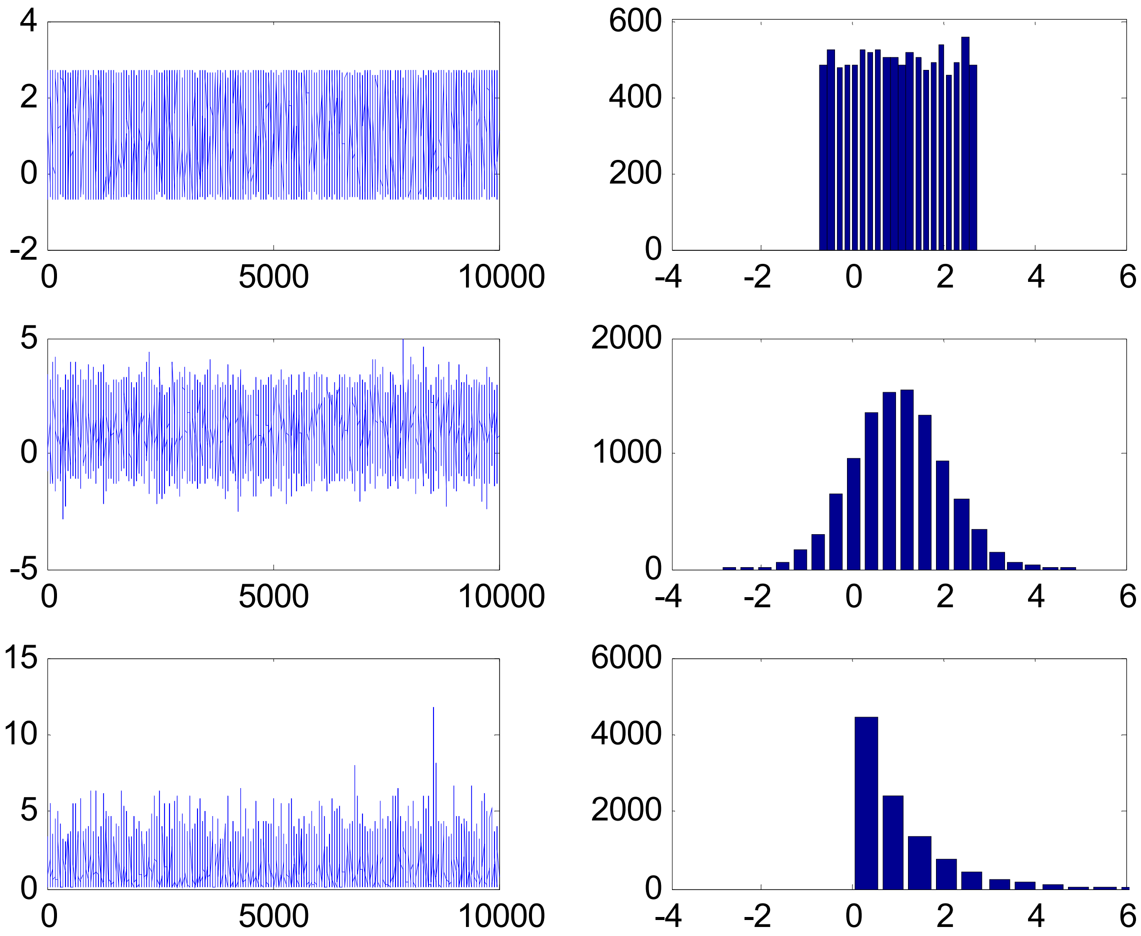
[IntroThreeContinuousRV.m]

Three Important Continuous RVs



Mean = 1
Std = 1
N = 1,000

Three Important Continuous RVs



Mean = 1

Std = 1

N = 10,000

Review: P[some condition(s) on X]

For discrete random variable,

8.13. Steps to find probability of the form P [some condition(s) on X] when the pmf $p_X(x)$ is known.

(a) Find the support of X.

(b) Consider only the x inside the support. Find all values of x that satisfy the condition(s).

(c) Evaluate the pmf at x found in the previous step.

(d) Add the pmf values from the previous step.

$$P[\text{some condition(s) on } X] = \sum p_X(x)$$

Discrete RV (X)

Sum over all the x values that satisfy the condition(s)

P[some condition(s) on X]

- **For discrete random variable,**

$$P[\text{some condition(s) on } X] = \sum \overbrace{p_X(x)}^{\text{probability mass function (pmf)}}$$

probability **mass** function (pmf)

Discrete RV

Sum over all the x values that satisfy the condition(s)

- **For continuous random variable,**

pmf → pdf

$\sum \rightarrow \int$

$$P[\text{some condition(s) on } X] = \int \overbrace{f_X(x)}^{\text{probability density function (pdf)}} dx$$

probability **density** function (pdf)

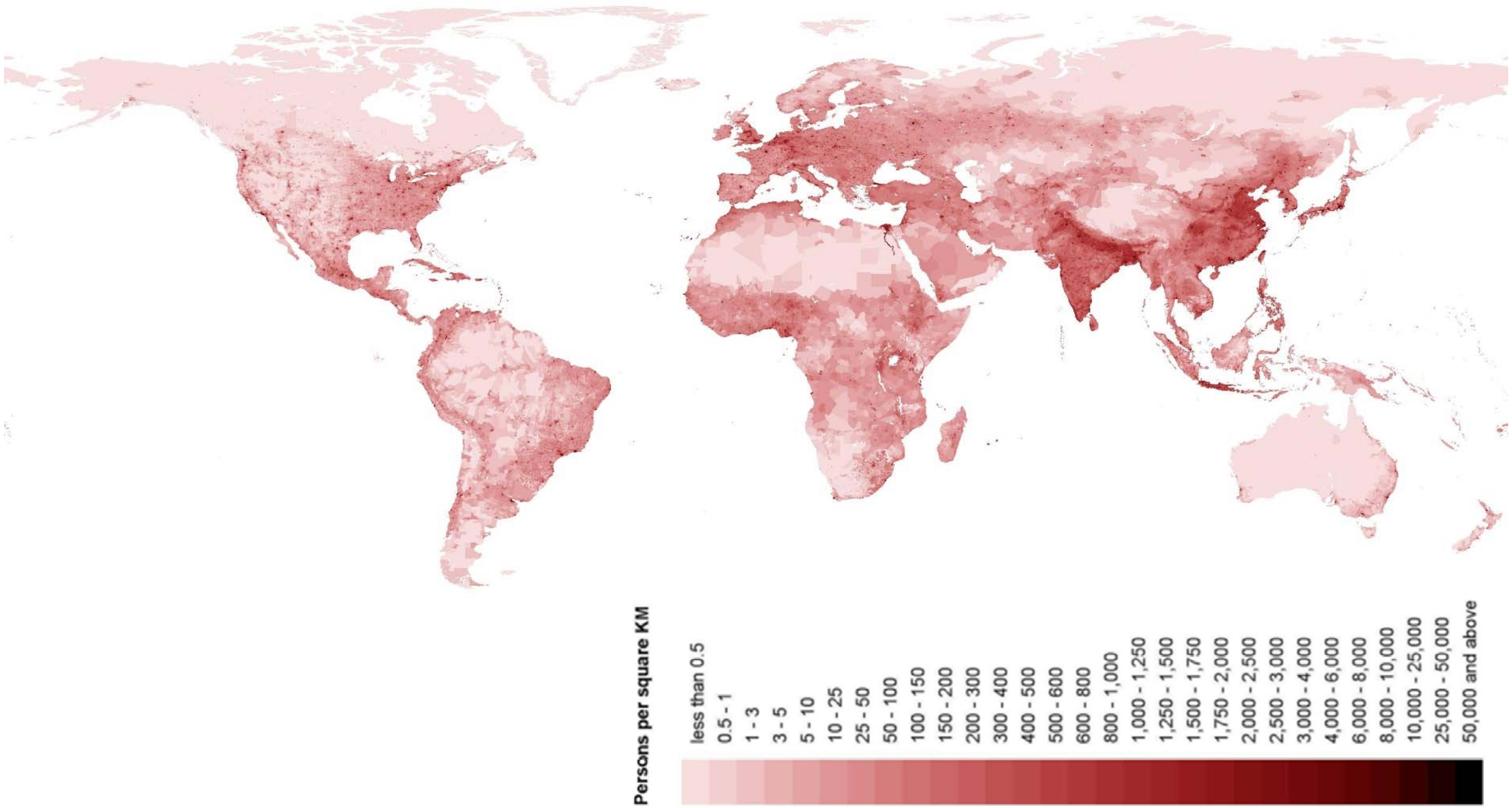
Continuous RV

Integrate over all the x values that satisfy the condition(s)

Support of a RV

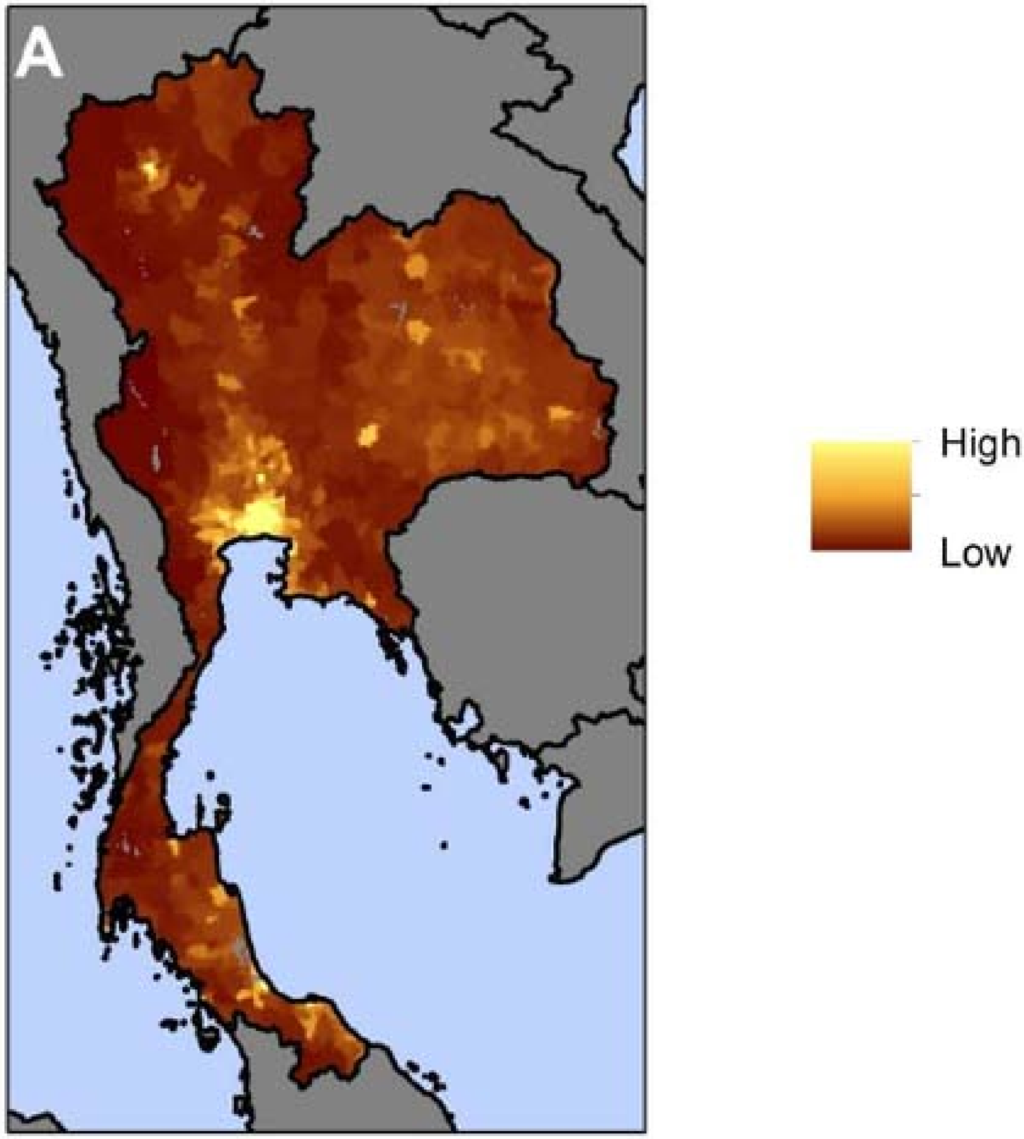
- In general, the **support** of a RV X is any set S such that
$$P[X \in S] = 1.$$
- In this class, we try to find the smallest (minimal) set that works as a support.
- **For discrete random variable,**
$$S_X = \{x: p_X(x) > 0\}$$
- **For continuous random variable,**
$$S_X = \{x: f_X(x) > 0\}$$

World Map of Population Density



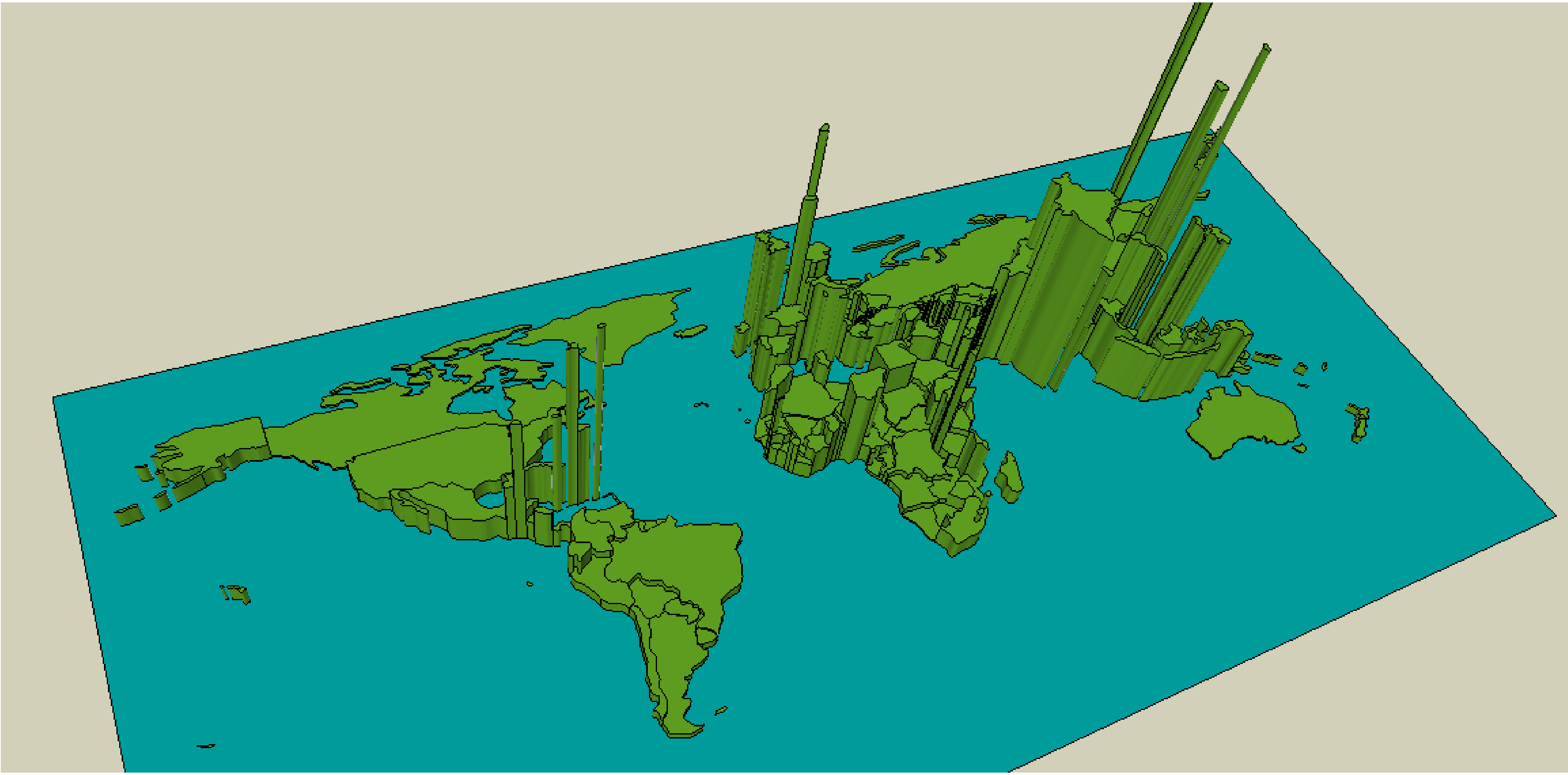
[http://i.imgur.com/gBYMfWO.jpg]

Thailand’s Population Density

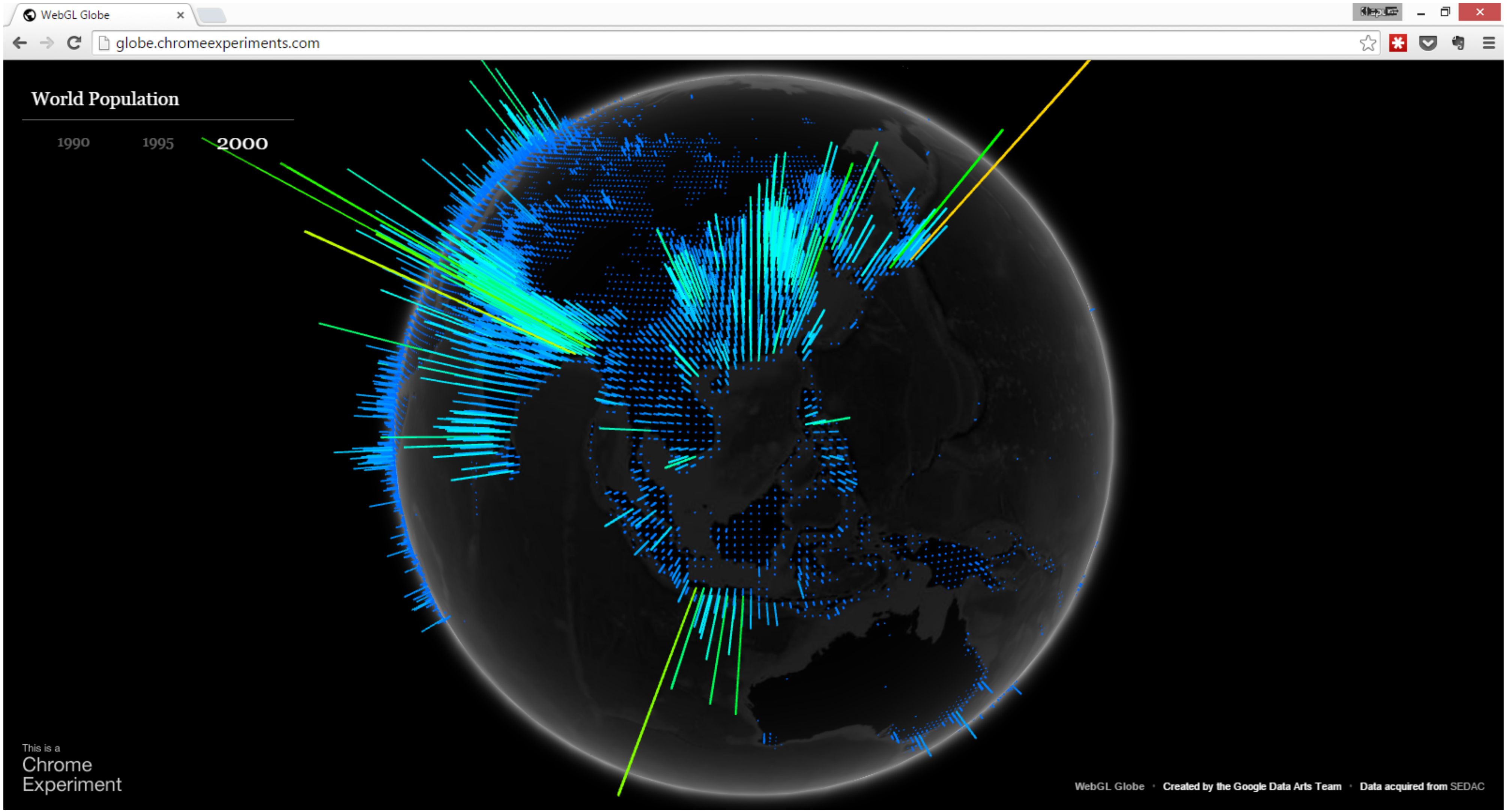


https://www.researchgate.net/publication/260378246_Climate-Related_Hazards_A_Method_for_Global_Assessment_of_Urban_and_Rural_Population_Exposure_to_Cyclones_Droughts_and_Floods/figures?lo=1

World Map of Population Density



World Map of Population Density



http://globe.chromeexperiments.com/

"Density"

- Density = quantity per unit of measure.
- Population Density = number of people per unit area
  - Location with high density value means there are a lot of people around that location.
  - Given a region, we integrate the density over that region to get the number of people residing in that region.
- Probability Density = probability per unit "length".
  - Given an interval, we integrate the density over that interval to get the probability that the RV will be in that interval.

References

- From Discrete to Continuous Random Variables: [Y&G] Sections 3.0 to 3.1
- PDF and CDF: [Y&G] Sections 3.1 to 3.2
- Expectation and Variance: [Y&G] Section 3.3
- Families of Continuous Random Variables: [Y&G] Sections 3.4 to 3.5

Course Outline

The following is a tentative list of topics with their corresponding chapters from the textb
Yates and Goodman. Each topic spans approximately one week.

1.	Introduction, Set Theory, Classical Probability	[1]
2.	Combinatorics: Four Principles and Four Kinds of Counting Problems	[1]
3.	Probability Foundations	[1]
4.	Event-based Conditional Probability	[1]
5.	Event-based Independence	[1]
6.	Random variables, Support, Probability Distribution	[2]
7.	**MIDTERM: 4 Oct 2018 TIME 09:00 - 11:00**	
8.	Discrete Random Variables	[2]
9.	Families of Discrete Random Variables and Introduction to Poisson Processes	[2,10]
10.	Real-Valued Functions of a Random Variable	[2]
11.	Expectation, Moment, Variance, Standard Deviation	[2]
12.	Continuous Random Variables	[3]
13.	Families of Continuous Random Variables and Introduction to Poisson Processes	[3,10]

- Excercise 15 Solution [Posted @ 4:30PM on N
- Excercise 16 Solution [Posted @ 3PM on Nov
- Slides [Posted @ 4:30PM on Nov 6]
- Part IV: Continuous Random Variables
  - Chapter 10 [Posted @ 10AM on Nov 5]
    - Annotated notes for Sections 10.1-10.3 [Pos
    - References
      - From Discrete to Continuous Random
      - PDF and CDF: [Y&G] Sections 3.1 to 3.2
      - Expectation and Variance: [Y&G] Sectio
      - Families of Continuous Random Variab
- Part V: Multiple Random Variables

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10.2 Properties of PDF and CDF

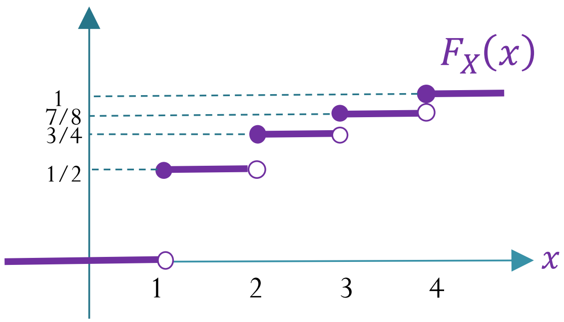
Sections 10.1-10.2

Discrete RV

- **pmf**: $p_X(x) \equiv P[X = x]$
  - Two characterizing properties:
    - $p_X(x) \geq 0$
    - $\sum_x p_X(x) = 1$
- $S_X = \{x: p_X(x) > 0\}$
- $P[\text{some condition(s) on } X]$

$$= \sum_{\substack{\{\text{all the } x \text{ values that} \\ \text{satisfy the condition(s)}\}}} p_X(x)$$

- **cdf** is a staircase function with jumps whose size at $x = c$ gives $P[X = c]$.

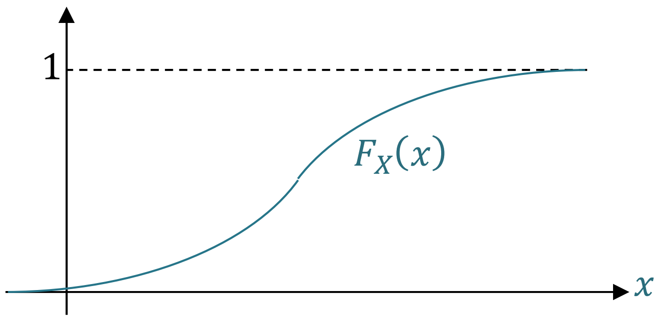


Continuous RV

- $P[X = x] = 0$
- **pdf**: $P[x_0 \leq x \leq x_0 + \Delta x] \approx \underbrace{f_X(x_0)}_{\text{probability per unit length}} \Delta x$
  - Two characterizing properties:
    - $f_X(x) \geq 0$
    - $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
- $S_X = \{x: f_X(x) > 0\}$
- $P[\text{some condition(s) on } X] =$

$$\int_{\substack{\{\text{all the } x \text{ values that} \\ \text{satisfy the condition(s)}\}}} f_X(x) dx$$

- **cdf** is a continuous function.



pdf and cdf for continuous RV

$P[a \le X \le b]$

$P[a < X < b]$

$P[a \le X < b]$

$P[a < X \le b]$

$P[a \le X \le b]$

$F_X(b) - F_X(a)$

$F_X(x) \equiv P[X \le x]$

$\text{“}\frac{d}{dx}\text{”} F_X(x)$

$\int_{-\infty}^{x} f_X(t)\,dt$

$\int_{a}^{b} f_X(x)\,dx$

$f_X(x)$

Finding Probabilities from CDF

Definition: $F_X(x) \equiv P[X \leq x]$

For **any RV**,	For **continuous RV**,
• $P[X \leq b] = F_X(b)$	• $P[X \leq b] = F_X(b)$ $P[X < b] = F_X(b)$
• $P[X > a] = 1 - F_X(a)$	• $P[X > a] = 1 - F_X(a)$ $P[X \geq a] = 1 - F_X(a)$
• $P[a < X \leq b] = F_X(b) - F_X(a)$	• $P[a < X \leq b] = F_X(b) - F_X(a)$ $P[a < X < b] = F_X(b) - F_X(a)$ $P[a \leq X < b] = F_X(b) - F_X(a)$ $P[a \leq X \leq b] = F_X(b) - F_X(a)$
• $P[X = a] = F_X(a) - F_X(a^-)$ (amount of jump in the CDF @ a)	• $P[X = a] = 0$

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10.3 Expectation and Variance

Chapter 9 vs. Section 10.3

Discrete RV	Continuous RV
$\mathbb{E}X = \sum_x x p_X(x)$	$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x) dx$
$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$
$\mathbb{E}[X^2] = \sum_x x^2 p_X(x)$	$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$

$$\text{Var}[X] = \mathbb{E}\left[(X - \mathbb{E}X)^2\right] = \mathbb{E}[X^2] - (\mathbb{E}X)^2$$

$$\sigma_X = \sqrt{\text{Var}[X]}$$

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10.4 Families of Continuous Random Variables

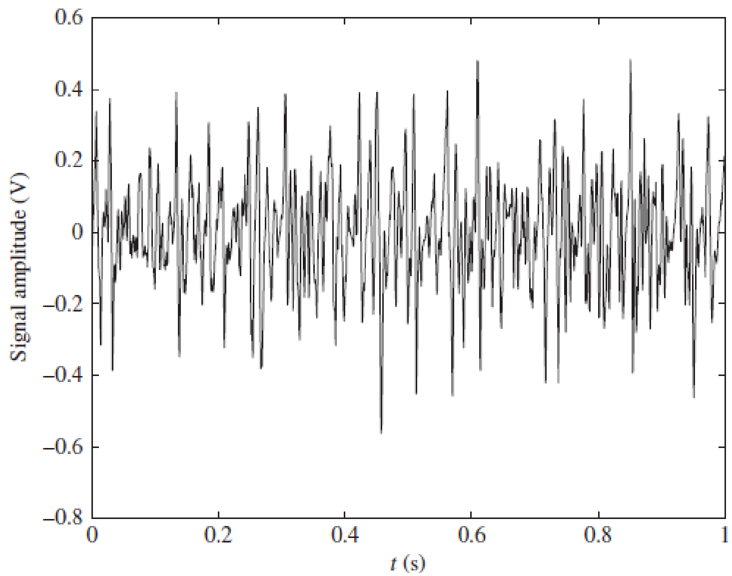
Johann Carl Friedrich Gauss

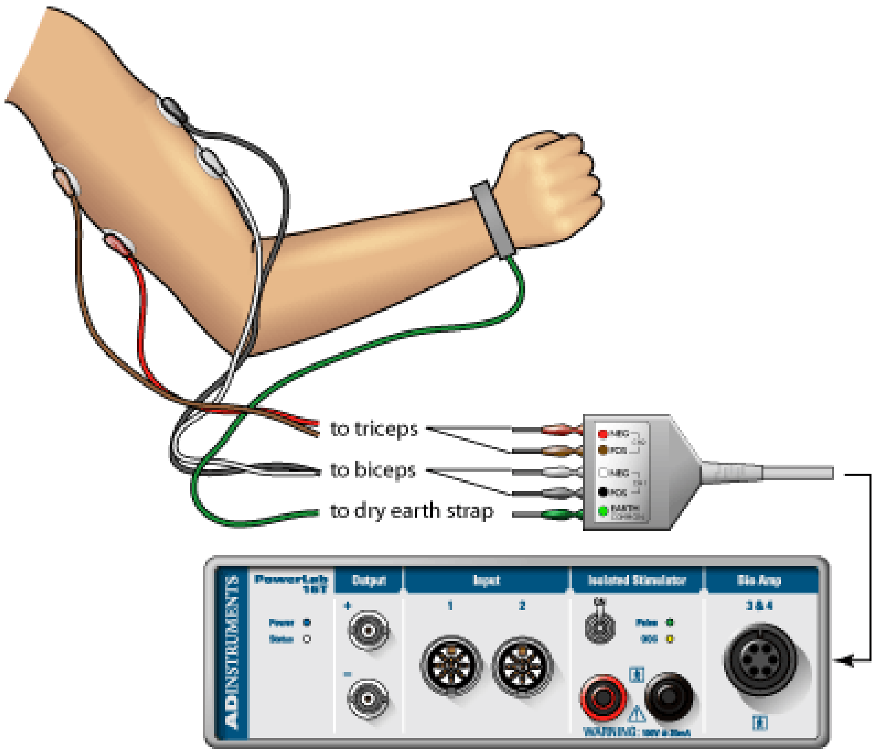
German 10-Deutsche Mark Banknote (1993; discontinued)

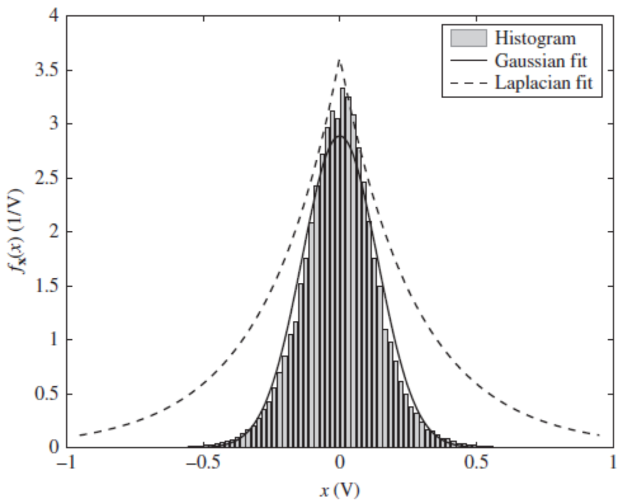
- 1777 –1855
- A German mathematician

Ex. Muscle Activity

- Look at electrical activity of skeletal muscle by recording a human electromyogram (EMG).



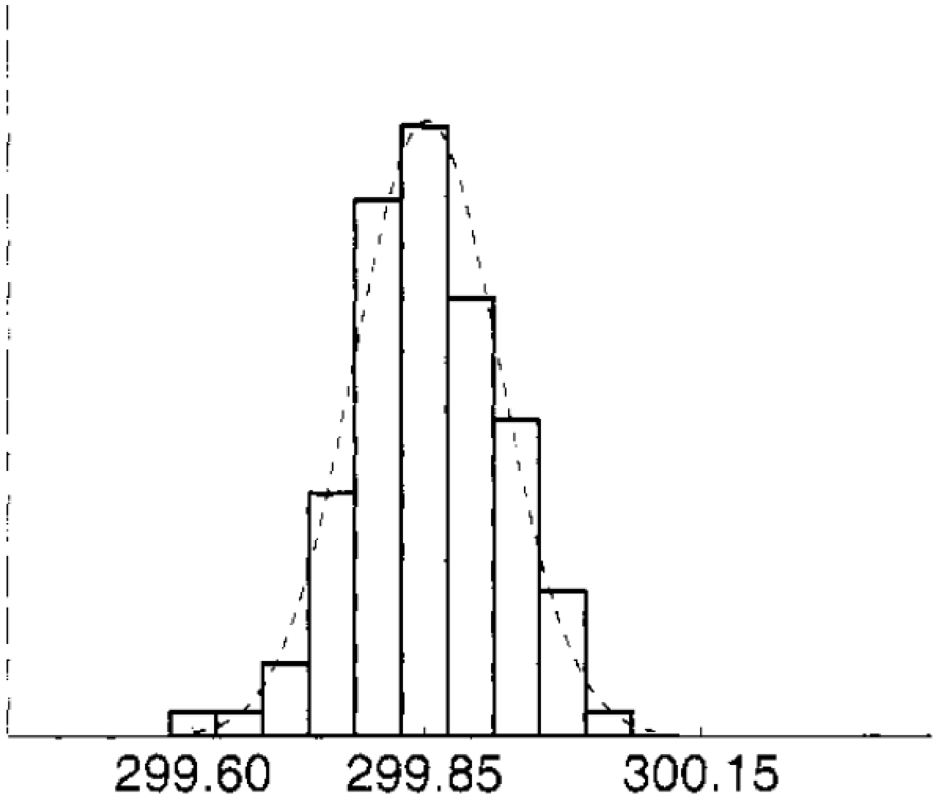




[http://www.adinstruments.com/solutions/education/ltexp/electromyography-emg-german]

Ex. Measuring the speed of light

- 100 measurements of the speed of light (×1,000 km/second), conducted by Albert Abraham Michelson in 1879.



Expected Value and Variance

"Proof" by MATLAB's symbolic calculation

```
>> syms x
>> syms m real
>> syms sigma positive
>> int(1/(sqrt(sym(2)*pi)*sigma)*exp(-(x-m)^2/(2*sigma^2)),x,-inf,inf)
ans =
1
>> EX = int(x/(sqrt(sym(2)*pi)*sigma)*exp(-(x-m)^2/(2*sigma^2)),x,-inf,inf)
EX =
m
>> EX2 = int(x^2/(sqrt(sym(2)*pi)*sigma)*exp(-(x-m)^2/(2*sigma^2)),x,-inf,inf)
EX2 =
-(2^(1/2)*(limit(- x*sigma^2*exp((x*m)/sigma^2 - m^2/(2*sigma^2) - x^2/(2*sigma^2)) - m*sigma^2*exp((x*m)/sigma^2 - m^2/(2*sigma^2) - x^2/(2*sigma^2)) -
(2^(1/2)*pi^(1/2)*sigma*erfi((2^(1/2)*(x - m)*i)/(2*sigma))*(m^2 + sigma^2)*i)/2, x == -Inf) - limit(- x*sigma^2*exp((x*m)/sigma^2 - m^2/(2*sigma^2) -
x^2/(2*sigma^2)) - m*sigma^2*exp((x*m)/sigma^2 - m^2/(2*sigma^2) - x^2/(2*sigma^2)) - (2^(1/2)*pi^(1/2)*sigma*erfi((2^(1/2)*(x - m)*i)/(2*sigma))*(m^2 +
sigma^2)*i)/2, x == Inf)))/(2*pi^(1/2)*sigma)
>> EX2 = simplify(EX2)
EX2 =
m^2 + sigma^2
>> VarX = EX2 - (EX)^2
VarX =
sigma^2
```